Spectral Representation at Finite Temperature

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Abstract

This is a short review on the thermal, spectral representation in the real-time version of the finite temperature quantum field theory. After presenting a clear derivation of the spectral representation, we discuss the properties of its spectral function. Two applications of this representation are then considered. One is the solution of the Dyson equation for the thermal propagator. The other is the formulation of the QCD sum rules at finite temperature.

1 Introduction

The real- and the imaginary- time versions of the quantum field theory in a medium (i.e., at finite temperature and/or density) have somewhat complementary virtues [1]. The real-time version is closer to the conventional (vacuum) field theory, involving no sum over frequencies and so requiring no analytic continuation to real energies. However, the price to pay for this closeness is the 2×2 matrix structure of all two-point correlation functions. Generally speaking, static thermodynamic quantities of a system are usually calculated in the imaginary-time version, while the real-time version appears convenient for calculating more detailed, particularly time dependent, quantities.

Here we are concerned with the real-time spectral representation for the thermal, two-point correlation functions of local operators. (If this operator is chosen to be just a field operator, the representation gives the corresponding thermal propagator.) Although the

real-time version is of rather recent development, such thermal spectral representations were obtained by Landau [2] as early as 1958, after Källen and Lehmann [3] derived them for the propagators in the vacuum. Earlier Low [4] had derived representations for non-vacuum matrix elements in the time component q_0 at fixed space component $\vec{q}$ of the 4-momentum variable q_μ, conjugate to the coordinate difference x^μ of the two operators in the matrix element. The Landau representation is of this variety, extending such matrix elements to their ensemble average.

Of course, Landau's derivation of the spectral representation in real-time was technically incomplete, as he did not take into account its 2×2 matrix structure. The complete representation was written by Semenoff and Umezawa [5] in 1983, after Umezawa and his collaborators [6] had established the real-time version.

In this work we derive in detail the spectral representation for the two-point correlation functions of local operators, obtaining the symmetry relations satisfied by the spectral function. To keep the kinematics simple we consider Lorentz scalar operators. As an example we calculate the spectral function to the leading order in scalar field theory. We go on to consider two applications of this spectral representation. One is to review the well-known reduction of the self-energy matrix to essentially a single function by using its factorized structure [7]. The other is to formulate the QCD sum rules in the real time, finite temperature field theory [8].

The derivation of the spectral representation is given in sec.2, giving an example of calculation of the spectral function in sec.3. In sec.4 we consider two applications, one to Dyson equation and the other to QCD sum rules. Finally in sec.5 we present a summary of the results derived in the paper.

2   Spectral Representation

We choose the contour in the complex time plane as originally proposed by Umezawa [6], to get a symmetric 2×2 matrix for the free thermal propagator. It consists effectively of two segments, one running along the real axis in the positive direction and the other parallel to it, but shifted by $-i\beta/2$, in the reverse direction, where β is the inverse

temperature T.

Consider the contour-ordered, two-point correlation function of a local operator. This operator may, in general, be a composite one, built out of fundamental fields, which may be scalar, spinor or vector fields. Thus in QCD the composite operator may be any one of the conserved currents, namely the vector current, $\bar{q}(x)\gamma_\mu\frac{\tau_i}{2}q(x)$ or the axial vector current, $\bar{q}(x)\gamma_\mu\gamma_5\frac{\tau_i}{2}q(x)$, where $q(x)$ is the doublet field of u and d quarks and τ_i's are the Pauli matrices. Besides such bosonic operators, one may also have fermionic operators like the so-called baryon currents. For the proton this operator is $\epsilon^{abc}[u^{aT}(x)C\gamma_\mu u^b(x)][\gamma^\mu\gamma^5 d^c(x)]_D$, where a, b, c are the colour indices, C is the charge conjugation matrix and D a Dirac index [9]. If the operator is not a composite one, but just one of the fundamental fields themselves, the correlation function becomes the thermal propagator for that field.

To avoid kinematic complications, we choose in the following a Lorentz scalar operator, which may be composite or fundamental and denote it by $O(x)$. Further we assume it to be bosonic. Then the contour-ordered two-point function may be put in the form of a 2×2 matrix,

$$\mathcal{T}(x-y) = i \begin{pmatrix} \langle TO(x)O(y)\rangle & \langle O(y-i\beta/2)O(x)\rangle \\ \langle O(x-i\beta/2)O(y)\rangle & \langle \overline{T}O(x-i\beta/2)O(y-i\beta/2)\rangle \end{pmatrix} , \qquad (2.1)$$

where T and $\overline{T}$ denote the usual time and anti-time ordering and $\langle\cdots\rangle$ denotes the ensemble average; thus for the operator O,

$$\langle O\rangle = Tr\, O e^{-\beta H}/Z, \quad Z = Tr\, e^{-\beta H} , \qquad (2.2)$$

H being the Hamiltonian of the system and Tr denoting trace over any complete set of states. In momentum space, the Fourier transform is denoted by the same symbol,

$$\mathcal{T}_{ab}(q) = \int d^4 z e^{iqz}\, \mathcal{T}_{ab}(z), \quad a, b = 1, 2 . \qquad (2.3)$$

Let us first consider the 11-component of the matrix. We evaluate the trace over a complete set of states $|m\rangle$, $m = 1, 2, ..$, which are eigenstates of the 4-momentum operator P_μ with eigenvalues $(p_m)_\mu$. Denoting $(p_m)_0$ by E_m, it becomes

$$\langle TO(x)O(y)\rangle = Z^{-1} \sum_m e^{-\beta E_m} \langle m|TO(x)O(y)|m\rangle , \qquad (2.4)$$

which is a sum over forward amplitudes weighted by the corresponding Boltzmann factors. Again inserting the same set of states to extract the co-ordinate dependence of the field operators, we get

$$
\langle TO(z)O(0)\rangle \;=\; Z^{-1}\sum_{m,n} e^{-\beta E_m}\left(\theta(z^0)e^{i(p_m-p_n)\cdot z}+\theta(-z^0)e^{-i(p_m-p_n)\cdot z}\right)
$$
$$
\times |\langle m|O(0)|n\rangle|^2 \; . \tag{2.5}
$$

It is now simple to work out the Fourier transform (2.3). The integration over space gives rise to δ-functions in 3-momentum, while that over the time variable produces the energy denominators. Inserting a δ-function in the energy variables, we may put it in the form

$$
\mathcal{T}_{11}(q) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}dq_0'\left(\frac{M^+(q_0',\vec{q})}{q_0-q_0'+i\epsilon}-\frac{M^-(q_0',\vec{q})}{q_0-q_0'-i\epsilon}\right) \;, \tag{2.6}
$$

where

$$
M^{\pm}(q_0,\vec{q}) = Z^{-1}\sum_{m,n}e^{-\beta E_m}(2\pi)^4\delta(q\pm p_m\mp p_n)|\langle m|O(0)|n\rangle|^2 \tag{2.7}
$$

The double sum over states may be converted back to the product of field operators giving

$$
M^+(q) = \int d^4z\, e^{iqz}\langle O(x)O(0)\rangle \;, \tag{2.8}
$$

and $M^-(q)$ having an identical expression with $O(x)$ and $O(0)$ interchanged.

In the case of the vacuum expectation value, the two spectral functions, which are functions of q^2 only, can be shown to be equal by the use of the causality requirement. However, in the present case, where they are functions of q_0 and $|\vec{q}|$ (or q^2 and $u\cdot q$ in a Lorentz covariant framework with u^μ being the four-velocity of the medium), a similar argument does not go through. But we still have two relations connecting $M^\pm$, namely the Kubo-Martin-Schwinger relation in momentum space,

$$
M^+(q_\mu) = e^{\beta q_0}M^-(q_\mu) \;, \tag{2.9}
$$

and the symmetry relation,

$$
M^+(-q_\mu) = M^-(q_\mu) \;. \tag{2.10}
$$

These relations are usually obtained from the operator representation (2.8). They may also be obtained from the double sum representation (2.7) : the relation (2.10) is evident,

while to get the relation (2.9) we interchange the dummy indices m, n in any one of $M^{\pm}(q)$ and use the δ-function to express $E_m - E_n$ by q_0.

Let us now define the spectral function ρ as

$$\rho(q_0, \vec{q}) \equiv M^+(q_0, \vec{q}) - M^-(q_0, \vec{q}) = \int d^4 z e^{iqz} \langle [O(x), O(0)] \rangle , \qquad (2.11)$$

which, on noting (2.10), is antisymmetric under $q_\mu \rightarrow -q_\mu$,

$$\rho(-q_\mu) = -\rho(q_\mu) . \qquad (2.12)$$

Also using (2.9), we can express both $M^{\pm}$ in terms of ρ

$$M^+(q_\mu) = \frac{e^{\beta q_0}}{e^{\beta q_0} - 1} \rho(q_\mu) ,$$

$$M^-(q_\mu) = \frac{1}{e^{\beta q_0} - 1} \rho(q_\mu) . \qquad (2.13)$$

We now wish to redefine the energy denominators in (2.6) with the Feynman $i\epsilon$ prescription. For this purpose we write it as,

$$\begin{aligned}
\mathcal{T}_{11}(q) &= i \int_{-\infty}^{\infty} \frac{dq_0'}{2\pi} \left[\frac{i\rho(q_0', \vec{q})}{q_0 - q_0' + iq_0'\epsilon} + \pi\delta(q_0 - q_0') \right. \\
&\quad \left. \times \{M^+(q_0') + M^-(q_0') - \text{sgn}(q_0')(M^+(q_0') - M^-(q_0'))\} \right]
\end{aligned} \qquad (2.14)$$

where the $\vec{q}$ dependence of $M^{\pm}$ is suppressed. Here we have written the integrals in (2.6) first as their principal values and then with the indicated $i\epsilon$ prescription, the terms with δ-functions serving to compensate these changes. Folding the range of integration on to $(0, \infty)$ and using the relations (2.12) and (2.13), we get the desired result *,

$$\mathcal{T}_{11}(q) = i \int_0^{\infty} \frac{dq_0'^2}{2\pi} \rho(q_0', \vec{q}) \left[\frac{i}{q_0^2 - q_0'^2 + i\epsilon} + 2\pi\delta(q_0'^2 - q_0^2) \frac{1}{e^{\beta|q_0|} - 1} \right] \qquad (2.15)$$

The 22-element of the matrix $\mathcal{T}$ may be simplified by invoking the translational invariance,

$$\mathcal{T}_{22}(x - y) = \theta(x^0 - y^0)\langle O(y)O(x) \rangle + \theta(y^0 - x^0)\langle O(x)O(y) \rangle \qquad (2.16)$$

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*The change of sign of $\vec{q}$ while using (2.12) is of no consequence here, since it occurs either as $\vec{q} \cdot \vec{q}$ or $\vec{q} \cdot \vec{p}$, where $\vec{p}$ is a 3-vector to be integrated out over its entire range.

Repeating the steps similar to above, we get for $\mathcal{T}_{22}(q)$ an expression identical to the one for $\mathcal{T}_{11}(q)$, except for complex conjugation of its first term in square bracket,

$$\mathcal{T}_{22}(q) = i \int_0^\infty \frac{dq_0'^2}{2\pi} \rho(q_0', \vec{q}) \left[\frac{-i}{q_0^2 - q_0'^2 - i\epsilon} + 2\pi\delta(q_0'^2 - q_0^2) \frac{1}{e^{\beta|q_0|} - 1} \right] \tag{2.17}$$

The 12- and 21- elements turn out to be identical,

$$\begin{aligned}
\mathcal{T}_{12}(q) = \mathcal{T}_{21}(q) &= i\frac{e^{\beta q_0/2}}{e^{\beta q_0} - 1} \rho(q_0, \vec{q}) \\
&= i \int dq_0'^2 \rho(q_0', \vec{q}) \delta(q_0'^2 - q_0^2) \frac{e^{\beta|q_0|/2}}{e^{\beta|q_0|} - 1} \tag{2.18}
\end{aligned}$$

Recognizing the density distribution function and the free propagator in vacuum with Feynman boundary condition,

$$n = \frac{1}{e^{\beta|q_0|} - 1} , \qquad \Delta = \frac{i}{q_0^2 - q_0'^2 + i\epsilon}$$

in the above expressions, we collect the results for the components as

$$\mathcal{T}_{ab}(q_0, \vec{q}) = \int_0^\infty \frac{dq_0'^2}{2\pi} \rho(q_0', \vec{q}) \mathcal{D}_{ab}^0(q_0, q_0') , \tag{2.19}$$

where $\mathcal{D}_{ab}^0$ is the free thermal propagator,

$$\begin{aligned}
\mathcal{D}_{ab}^0 &= i \begin{pmatrix} (1+n)\Delta + n\Delta^* & \sqrt{n(1+n)}(\Delta + \Delta^*) \\ \sqrt{n(1+n)}(\Delta + \Delta^*) & n\Delta + (1+n)\Delta^* \end{pmatrix} \tag{2.20} \\
&= U(|q_0|) \, i \begin{pmatrix} \Delta & 0 \\ 0 & \Delta^* \end{pmatrix} U(|q_0|) , \tag{2.21}
\end{aligned}$$

with

$$U = \begin{pmatrix} \sqrt{1+n} & \sqrt{n} \\ \sqrt{n} & \sqrt{1+n} \end{pmatrix} . \tag{2.22}$$

The matrix U does not depend on the integration variable in (2.19) and like the free propagator, the correlation function also factorizes,

$$\mathcal{T}_{ab}(q_0, \vec{q}) = U(|q_0|) \begin{pmatrix} T & 0 \\ 0 & -T^* \end{pmatrix} U(|q_0|) \tag{2.23}$$

where

$$T(q_0, \vec{q}) = \int_0^\infty \frac{dq_0'^2}{2\pi} \frac{\rho(q_0', \vec{q})}{q_0'^2 - q_0^2 - i\epsilon} \tag{2.24}$$

Thus the matrix function is given essentially by a single analytic function $T(q_0, \vec{q})$. This integral representation, as such, may not converge, when it would need subtractions. However, this point does not concern us in the following.

As a simple example, let us calculate $T(q_0, \vec{q})$ for the case where the operator $O(x)$ is a scalar field operator $\phi(x)$, representing particles of mass m. Then from Eq.(2.11) we get immediately the spectral function as

$$\rho(q_0, \vec{q}) = 2\pi \ \text{sgn}(q_0)\delta(q_0^2 - \vec{q}^{\,2} - m^2)$$

giving

$$T(q) = \frac{-1}{q^2 - m^2 + i\epsilon} \tag{2.25}$$

which is the free propagator function for the scalar field with Feynman boundary conditions.[†]

The important point to notice here is that there is only one spectral function ρ giving all the four components of the correlation function and it may be obtained by calculating any one of the components, say $\mathcal{T}_{11}$. But although $\rho \equiv (M^+ - M^-)$ is (twice) the imaginary part of T, it is not so for $\mathcal{T}_{11}$: As seen from (2.6), (twice) the imaginary part of $\mathcal{T}_{11}$ is $(M^+ + M^-)$. The two, however, are related through eq.(2.13),

$$\rho(q_0, \vec{q}) = 2 \tanh(\beta q_0/2) \ Im\mathcal{T}_{11}(q_0, \vec{q}) \tag{2.26}$$

Their real parts are, however, equal,

$$ReT(q_0, \vec{q}) = Re\mathcal{T}_{11}(q_0, \vec{q}) \ . \tag{2.27}$$

Eqs.(2.23 - 24) constitute the finite temperature extension of the Källan-Lehmann representation in vacuum. The breakdown of Lorentz invariance is explicit as it treats $\vec{q}$ as fixed. (As already mentioned, it can be formally restored, but only by introducing the four-velocity of the heat bath.) Note the relations (2.26-27) between (the real and imaginary parts of) T and the $\mathcal{T}_{11}$- component. Such relations also exist among the components of the self-energy matrix, to be discussed below.

[†]The occurrence of -1 instead of i in the numerator of Eq. (2.25) is due to the presence of i in the definition of (2.1) of the correlation function.

Before leaving this section, we wish to point out that an equation like (2.7) giving the spectral function as a double sum over states is not very convenient for its evaluation. For, the δ-function in it seems to indicate that even initial and final states with a large number m and n of particles would give a contribution in the low energy region provided their momentum difference $(p_m - p_n)$ is small. Actually most of these particles are from the heat bath and do not participate in the interaction with the operators in the two point function. Thus it is much simpler to calculate the spectral function from the relevant Feynman diagrams, where the thermal propagator properly takes into account the particles of the heat bath in the distribution function present in it. Accordingly, we turn to the Feynman diagrams in the next section to calculate the spectral function.

3 An Example

Let us evaluate the correlation function perturbatively for the case where

$$O(x) = \phi_1(x)\phi_2(x) \quad , \tag{3.1}$$

the fields $\phi_1(x)$ and $\phi_2(x)$ being two different scalar fields representing particles of masses m_1 and m_2. Then the perturbative evaluation for the 11-component follows from

$$T_{11}(q) = i \int d^4x \; e^{iqx} \; \langle \, Te^{i \int \mathcal{L}_{int}(y)d^4y}\phi_1(x)\phi_2(x)\phi_1(0)\phi_2(0) \, \rangle \;, \tag{3.2}$$

where $\mathcal{L}_{int}(\phi_1, \phi_2)$ is the interaction part of the Lagrangian involving $\phi_1(x)$ and $\phi_2(x)$ (Fig.1). Usual Feynman rules apply with the vacuum propagators being replaced by the thermal ones,

$$\int d^4x e^{iqx}\langle \, T\phi_i(x)\phi_i(0) \, \rangle = \frac{i}{k^2 - m_i^2} + 2\pi\delta(k^2 - m_i^2)n_i(k), \quad i = 1,2 \tag{3.3}$$

where $n_i(k)$ is the distribution function of particle type i. To leading order the 11-component is given by

$$\mathcal{D}_{11}(q) = -i \int \frac{d^4k}{(2\pi)^4}\{(1 + n_1)\Delta_1 + n_1\Delta_1^*\}\{(1 + n_2)\Delta_2 + n_2\Delta_2^*\} \tag{3.4}$$
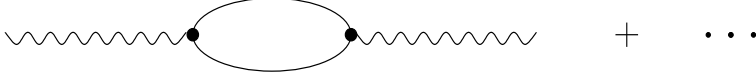
Figure 1: Two point function for $O(x) = \phi_1(x)\phi_2(x)$. The dots indicate higher order diagrams, whose nature and contribution depend on $\mathcal{L}_{int}(x)$

where

$$n_1 = \frac{1}{e^{\beta w_1} - 1}, \qquad w_1 = \sqrt{\vec{k}^2 + m_1^2}$$

$$n_2 = \frac{1}{e^{\beta w_2} - 1}, \qquad w_2 = \sqrt{\left(\vec{k} - \vec{q}\right)^2 + m_2^2}$$

$$\Delta_1 = \frac{i}{k^2 - m_1^2 + i\epsilon}, \qquad \Delta_2 = \frac{i}{(k-q)^2 - m_2^2 + i\epsilon} \qquad (3.5)$$

Each term in the the integrand of (3.4) has a product of two propagators. Their singularities in k_0 are only due to the poles in these propagators. So the integration over k_0 is performed by closing the contour in the k_o-plane ‡, after which the imaginary part may easily be read off,

$$
\begin{aligned}
Im\mathcal{D}_{11} \;=\; & \pi \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{4w_1 w_2} \\
& \Big[\{(1 + n_1)(1 + n_2) + n_1 n_2\}\{\delta(q_0 - w_1 - w_2) + \delta(q_0 + w_1 + w_2)\} \\
& + \{(1 + n_1)n_2 + (1 + n_2)n_1\}\{\delta(q_0 - w_1 + w_2) + \delta(q_0 + w_1 - w_2)\} \Big]
\end{aligned}
$$

$$(3.6)$$

It will be noted that the factors involving the density distribution functions are not appropriate for the interpretation in terms of emission and absorption probabilities of the particles [10]. The desired interpretation follows, if we use the energy conserving δ-functions to rewrite it as

$$Im\mathcal{D}_{11} = \coth(\beta q_0/2)\ I\ , \qquad (3.7)$$

where

$$
\begin{aligned}
I(q_0, \vec{q}) \;=\; & \pi \int \frac{d^3\vec{k}}{(2\pi)^3 4 w_1 w_2} \Big[(1 + n_1 + n_2)\{\delta(q_0 - w_1 - w_2) - \delta(q_0 + w_1 + w_2)\} \\
& + (n_2 - n_1)\{\delta(q_0 - w_1 + w_2) - \delta(q_0 + w_1 - w_2)\} \Big]\ .
\end{aligned}
$$

$$(3.8)$$

---

‡By power counting the integral in (3.2) is divergent. But the divergence resides only in the real part.

Thus from (2.26) and (3.5) we see that the tanh- and coth- factors cancel out in the spectral function giving

$$\rho(q_0, \vec{q}\,) = 2I(q_0, \vec{q}\,) \ , \tag{3.9}$$

which agrees with the one obtained from the imaginary-time formulation [10]. Also notice that $I(q_0, \vec{q}\,)$ is antisymmetric under $q_0 \to -q_0$ in agreement with Eq.(2.12).

Although we are discussing in this work only the bosonic propagator and bosonic intermediate states in its spectral representation, the cancellation of trigonometric functions is quite general. As long as the propagator is bosonic, there arises a coth- factor in the spectral function, even if the intermediate state is a fermion-antifermion system. But if we consider a fermionic propagator, the tanh- and coth- factors interchange in the expressions analogous to (2.26) and (3.7). Thus the trigonometric factors cancel in the spectral representation in all cases. Also we note that if there is a chemical potential μ in the Boltzmann factor, the argument, $\beta q_0/2$ of the trigonometric factors are replaced by $\beta(q_0 - \mu)/2$, besides other changes.

4   Applications

A. Dyson Equation

The spectral representation is very general in that it is valid for all q^2. If, however, we are interested only in the immediate neighbourhood of the pole in the propagator, the Dyson equation is the appropriate tool to work with (Fig.2). But even here the factorizibility of the full propagator, as established by the spectral representation, may be used to show that the self-energy matrix appearing in the Dyson equation has actually a simple structure, reducing essentially to a single function.

At finite temperature the Dyson equation for the propagator matrix in momentum space is

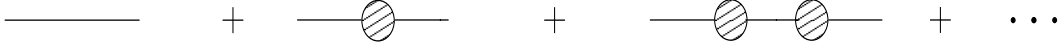$$\mathcal{D} = \mathcal{D}_0 - \mathcal{D}_0 \Sigma \mathcal{D} \ , \tag{4.1}$$

Figure 2: The series of diagrams summed up in the Dyson equation for the propagator

where Σ is the self-energy matrix. It has the solution

$$\mathcal{D}^{-1} = \mathcal{D}_0^{-1} + \Sigma \tag{4.2}$$

Using the factorized forms for the propagators, it gives

$$\begin{pmatrix} 1/D & 0 \\ 0 & -1/D^* \end{pmatrix} = \begin{pmatrix} -(q^2 - m^2) & 0 \\ 0 & q^2 - m^2 \end{pmatrix} + U\Sigma U \tag{4.3}$$

It follows that Σ must be of the form

$$\Sigma(q) = U^{-1}(q_0) \begin{pmatrix} \tilde{\Sigma} & 0 \\ 0 & -\tilde{\Sigma}^* \end{pmatrix} U^{-1}(q_0) \tag{4.4}$$

where $\tilde{\Sigma}(q)$ is the self-energy function. Then we have the solution

$$D = -\frac{1}{q^2 - m^2 - \tilde{\Sigma} + i\epsilon} \tag{4.5}$$

All the components of the self-energy matrix can now be expressed through the function $\tilde{\Sigma}$. From (4.4) we get [7]

$$\Sigma_{11} = -\Sigma_{22}^* = Re\tilde{\Sigma} + i(1 + 2n)Im\tilde{\Sigma}$$

$$\Sigma_{12} = \Sigma_{21} = -2i\sqrt{n(1+n)}Im\tilde{\Sigma} \tag{4.6}$$

These relations should not be interpreted as constraints imposed on the elements of the Σ-matrix by the solution of the Dyson equation. They are, in fact, automatically satisfied by the expressions obtained from perturbation theory. The above method of solution is just a convenient way to establish them.

Let us note here that the factorization (4.3) of the *full* matrix propagator is really not necessary to solve the Dyson equation. Indeed, writing $\Sigma = U^{-1}\Sigma'U^{-1}$ and using the factorization of the *free* propagator alone, eqn.(4.2) gives

$$\mathcal{D} = U \begin{pmatrix} -(q^2 - m^2) + \Sigma'_{11} & \Sigma'_{12} \\ \Sigma'_{21} & (q^2 - m^2) + \Sigma'_{22} \end{pmatrix}^{-1} U \tag{4.7}$$

On obtaining the inverse, one finds that each of the components of the propagator matrix is a linear combination of the pole terms, $(q^2 - m^2 - \Sigma_+)^{-1}$ and $(q^2 - m^2 - \Sigma_-)^{-1}$, where

$$\Sigma_\pm = \frac{1}{2}(\Sigma'_{11} - \Sigma'_{22}) \pm \frac{1}{2}\left((\Sigma'_{11} + \Sigma'_{22})^2 - 4\Sigma'_{12}\Sigma'_{21}\right)^{1/2} . \tag{4.8}$$

Thus although we can solve Dyson's equation, we cannot relate the components of Σ to a single function $\tilde{\Sigma}$, without using factorizibility of the propagator. But on using these relations, we see that $\Sigma_\pm$ indeed reduces respectively to $\tilde{\Sigma}$ and $\tilde{\Sigma}^*$.

The two-particle contribution to the spectral function considered in sec.3 may also be taken as a contribution to the self-energy. Thus the calculation of $\mathcal{D}_{11}$ there also serves as an example of Σ_{11}.

B. Thermal QCD Sum Rules

A vacuum QCD sum rule may be derived for a two-point correlation function, $\langle 0|T\mathcal{O}_1(x)\mathcal{O}_2(0)|0\rangle$ of any two operators $\mathcal{O}_1(x)$ and $\mathcal{O}_2(x)$, which can be built out of quark and gluon fields of the QCD theory. As already mentioned, these operators can be mesonic, like the conserved currents of the flavour $SU(2)$ group, or can be fermionic, like the baryon currents. The QCD sum rules [11] are obtained by equating the Fourier transforms of the spectral representation to that of the Operator Product Expansion (OPE) of the two-point function at a convenient space-like momentum.

These vacuum sum rules have already been extended to the medium at finite temperature (with or without chemical potential) [8, 12]. It is not our purpose here to review this topic in detail. Instead, we merely show how one deals with the 2×2 matrix structure of the two-point function in the formulation of these sum rules.

To bring out the essential features of these thermal QCD sum rules without getting involved in kinematic complications, we again consider the two-point correlation function (2.1) of a bosonic Lorentz scalar operator $O(x)$, satisfying the spectral representation (2.23). Further, if we take q^2 space-like, $q^2 = q_0^2 - \vec{q}^{\,2} = -Q^2, Q^2 > 0$, where the function $T(Q^2, \vec{q}^{\,2})$ is real, it reduces to

$$\mathcal{T}_{ab}(Q^2, \vec{q}^{\,2}) = T(Q^2, \vec{q}^{\,2}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.9}$$

where

$$T(Q^2, \vec{q}^{\,2}) = \int_{-\vec{q}^{\,2}}^{\infty} \frac{dq^2}{2\pi} \frac{\rho(q^2, \vec{q}^{\,2})}{q^2 + Q^2} \qquad (4.10)$$

with ρ given by (2.26). But as we already pointed out at the end of sec.2, it is more useful to calculate it perturbatively within the context of an appropriate effective field theory.

It is now clear that the 11- (or 22-) component of the matrix (4.9) contains all the information in the space-like region. We may then obtain the thermal spectral sum rules by considering this component alone.

The other ingredient needed to write the sum rules is the OPE for the 11-component. It is in deriving this expansion that the actual quark and/or gluon structure of the operator $O(x)$ must be spelt out. What the OPE does is to expand the two-point function in terms of local operators having finite matrix elements, its singular behaviour at short distance being segregated in c number (Wilson) coefficient functions.[§] At sufficiently high Q^2, its Fourier transform may be written as,

$$i \int d^4 x e^{iqx} \langle TO(x)O(0) \rangle_{11} = \sum c_i \frac{\langle O_i \rangle}{(Q^2)^m} \ , \qquad (4.11)$$

where c_i are numbers. The local operators O_i are labelled by the index i denoting their dimensions. The inverse powers of Q^2 are determined by dimensional counting, $m = i/2 + 2 - p$, where p is the dimension of the operator $O(x)$.

The sum rule follows from equating the spectral representation (4.10) to the OPE (4.11) in a convenient region of Q^2. After Borel improvement [11], it finally becomes

$$\frac{1}{2\pi M^2} \int_{-|\vec{q}|^2}^{\infty} ds \ e^{-s/M^2} \rho(s, \vec{q}^2) = \sum c_i \frac{\langle O_i \rangle}{(m-1)!(M^2)^m} \ , \qquad (4.12)$$

where M is the Borel mass replacing Q.

In writing sum rules of practical interest, we have, of course, to deal with more complicated kinematics depending on the nature of the operators $O(x)$ in the two-point function. But it is clear that the method of derivation and the structure of these sum

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[§] The off-diagonal elements are actually regular as the two arguments of the operator $\mathcal{O}$ cannot tend to each other for finite β.

rules will be the same as in the above illustrative example. The special features of these sum rules over the corresponding vacuum ones and their evaluations are described in the literature [12, 13, 14].

5   Conclusion

We present a detailed derivation of the spectral representation and discuss properties of the spectral function. We calculate the spectral function, as a proto-type example, in a scalar field theory and note its agreement with the expression obtained from the imaginary time version of the finite temperature field theory. Relationships similar to those among the components of the self-energies are shown to exist also among the components of the two-point correlation matrix.

We then review two important applications of this representation. One is the solution of the Dyson equation for the thermal propagator. The factorizability of the full propagator (as well as of the free one) leads to a simple solution, relating at the same time all the four components of the self-energy matrix to a single function. We also point out that the Dyson equation could be solved even without this input of factorizability, but then the simple form of the self-energy matrix could not be exposed. The latter could, of course, be inferred from its perturbative calculation in each order.

The other application relates to the formulation of the QCD sum rules in the real time thermal field theory. We show how the matrix structure of the two-point function simplifies at space-like momenta, where it is needed for writing the sum rules. We can then work with its 11- component, obtaining the familiar sum rules written in the literature.

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